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Exact solution of the Ising model on a pentagonal lattice

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Abstract
It is shown that the Ising model on a two-dimensional lattice with pentagonal tiling and nearest-neighbour interactions is equivalent to the Ising model on the Union-Jack lattice with nearest- and second nearest-neighbour non-crossing interactions. Besides the critical temperature, exact expressions are obtained for the spontaneous magnetization for both types of lattice sites.

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The exact solution of the planar Ising model on the square lattice with nearest-neighbour interactions obtained by Onsager almost six decades ago has a prominent place in the history of statistical mechanics (Lieb 1997). The following years witnessed new exact results for more complex lattices (Syozi 1972, Baxter 1982), including interactions with further neighbours (Vaks *et al* 1965) or triplet interactions (Baxter and Wu 1973a, 1973b) among other extensions. Nevertheless it is still of interest to find the exact critical point and other statistical properties for specific lattices with particular interactions. Recently, Oitmaa and Keppert (2002) obtained exact expressions for the critical temperature and the magnetization for the 4–6 lattice which represents a particular type of tiling the plane with squares and hexagons. A guide to the literature on phase transitions and critical phenomena was prepared by Tobochnik (2001).

In the following we consider an Ising model on a planar lattice (see figure 1) where the tiling is achieved with one type of polygons, namely pentagons, and which appears not to have been dealt with in the literature. The lattice may be viewed as a pattern with chessboard ordering with the elementary motif rotated by 90° in the neighbouring square plaquettes. There are two types of lattice sites with different coordination numbers equal to 3 and 4. First we derive an exact relationship from which the critical temperature can be computed and then obtain the order parameter for both types of lattice sites. The method used in the analysis is based on decoration–iteration and star–triangle transformations commonly used in Ising model studies (Fisher 1959, Syozi 1972).

To introduce the notation let us consider the elementary square plaquette depicted in figure 2. The letters a, b, c, d are used for the spins located at the fourfold coordinated sites in the corners, while $s_{1,2}$ represent the internal spins with threefold coordination. There are two

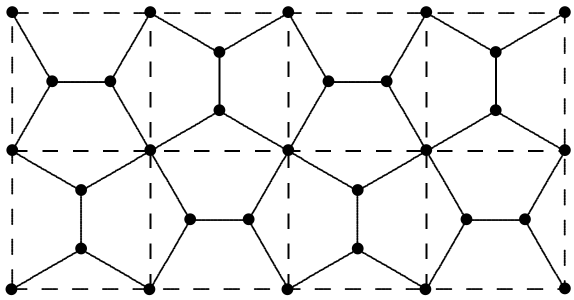

Figure 1. Pentagonal lattice of dark circles with interaction between nearest-neighbour sites. The dashed lines indicate the underlying square structure.

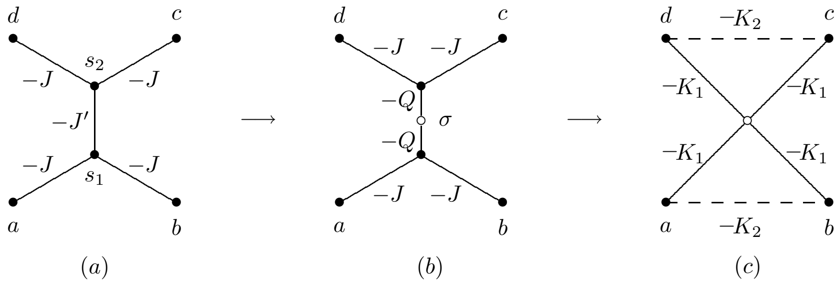

Figure 2. An elementary plaquette (*a*) and its subsequent transformations.

distinct interaction parameters, J and J', coupling particular nearest-neighbour pairs of spins (figure 2(*a*)). The contribution to the Hamiltonian from an elementary plaquette is given by

$$H = -J[s_1(a+b) + s_2(c+d)] - J's_1s_2. \tag{1}$$

As a first step in the transformation an additional spin σ is introduced at each internal bond (figure 2(*b*)) using the identity

$$\mathrm{e}^{K's_1s_2} = A\sum_{\sigma=\pm1}\mathrm{e}^{Q\sigma(s_1+s_2)} = 2A\cosh Q(s_1+s_2) \tag{2}$$

which is valid for all possible combinations of $s_{1,2} = \pm1$. Here $K' = \beta J'$, $\beta = 1/k_BT$, $A = \mathrm{e}^{-K'}/2$ and Q, the effective interaction between the spins σ and $s_{1,2}$, is defined by

$$\cosh 2Q = \mathrm{e}^{2K'}. \tag{3}$$

The validity of this relationship is restricted to the case $K' > 0$.

The next step is to perform the summation in the partition function over the internal $s_{1,2}$-spins. This is the well-known star–triangle transformation which when applied as a result gives the effective interactions dipicted in figure 2(*c*). The magnitudes of the effective interactions follow from the expression

$$\sum_{s_1=\pm1}\mathrm{e}^{[Q\sigma+K(a+b)]s_1} = 2\cosh[Q\sigma + K(a+b)] = B\mathrm{e}^{K_1\sigma(a+b)+K_2ab} \tag{4}$$

where $K = \beta J$, $\sigma = \pm1$, $a = \pm1$, $b = \pm1$ and

$$\mathrm{e}^{4K_1} = \frac{\cosh(Q+2K)}{\cosh(Q-2K)} \tag{5}$$

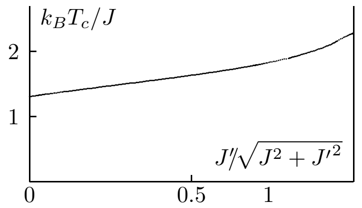

Figure 3. The reduced critical temperature k_BT_c/J versus $J/\sqrt{J^2+J'^2}$.

$$\mathrm{e}^{4K_2} = \frac{\cosh(Q+2K)\cosh(Q-2K)}{\cosh^2 Q} = \frac{\cosh 2Q + \cosh 4K}{\cosh 2Q + 1}. \tag{6}$$

The prefactor $B^4 = 16\cosh^2 Q \cosh(Q+2K)\cosh(Q-2K)$ does not enter any of the subsequent expressions.

As a result of the transformations and the structure of the initial pentagonal lattice, we end up with an Ising model on the Union-Jack lattice with isotropic nearest-neighbour interactions defined by K_1 and non-crossing diagonal interactions between the second nearest neighbours given by K_2. The Ising model on the Union-Jack lattice has been solved (Vaks *et al* 1965) and additional exact results were obtained by others (Choy and Baxter 1987). Using their results, the critical point of our model can be obtained from the equation

$$k = 1 \tag{7}$$

where

$$k = \frac{4x^2y^2(1+x^2)^2}{(1-x^4)^2 + 4x^4(1-y^2)^2} \tag{8}$$

and

$$x = \mathrm{e}^{-2K_1} \qquad y = \mathrm{e}^{-2K_2}. \tag{9}$$

Furthermore, the magnetization $M_0 = \langle a \rangle$ of the corner spins is described by

$$M_0 = (1-k^2)^{1/8}. \tag{10}$$

The dependence of the critical temperature as a function of the relative strength of the coupling parameters is illustrated in figure 3.

To proceed with the evaluation of M_s, two identities will be useful,

$$\tanh[Q\sigma + K(a+b)] = A\sigma + B(a+b) + Cab\sigma \tag{11}$$

and

$$\tanh[K_1(a+b+c+d)] = D(a+b+c+d) + E(abc+bcd+cda+dab). \tag{12}$$

The coefficients in (11) and (12) are uniquely determined from the equations obtained by substitution of all possible combinations of $\sigma, a, b, c, d = \pm 1$. They yield

$$\begin{aligned} A &= \tfrac{1}{4}[\tanh(Q+2K) + \tanh(Q-2K) + 2\tanh Q] \\ B &= \tfrac{1}{4}[\tanh(Q+2K) - \tanh(Q-2K)] \\ C &= \tfrac{1}{4}[\tanh(Q+2K) + \tanh(Q-2K) - 2\tanh Q] \end{aligned} \tag{13}$$

$$D = \frac{\tanh 4K_1 + 2\tanh 2K_1}{8} \qquad E = \frac{\tanh 4K_1 - 2\tanh 2K_1}{8}. \tag{14}$$

Table 1. Magnetization M_0 and M_s for $J'/\sqrt{J^2 + J'^2} = 0.1$ and $k_B T_c/J = 1.377\,454\,026$.

T/T_c	0	0.5	0.8	0.95	0.99	1
$M_0(T)$	1	0.9998	0.9780	0.8711	0.7256	0
$M_s(T)$	1	0.9972	0.9554	0.8393	0.6968	0

Table 2. Magnetization M_0 and M_s for $J'/\sqrt{J^2 + J'^2} = 1/\sqrt{2}$ and $k_B T_c/J = 1.799\,166\,700$.

T/T_c	0	0.5	0.8	0.95	0.99	1
$M_0(T)$	1	0.9996	0.9726	0.8609	0.7140	0
$M_s(T)$	1	0.9986	0.9696	0.8644	0.7195	0

Returning to the evaluation of the magnetization of the internal spins s_i, $i = 1, 2$, and using (11), we obtain an equation that relates $M_s = \langle s_1 \rangle$ to $M_0 = \langle a \rangle$, $M_\sigma = \langle \sigma \rangle$ and the three-spin correlation $\langle ab\sigma \rangle$,

$$\langle s_1 \rangle = \langle \tanh[Q\sigma + K(a+b)] \rangle = A\langle \sigma \rangle + 2B\langle a \rangle + C\langle ab\sigma \rangle. \tag{15}$$

The magnetization of the internal spins on the Union-Jack lattice M_σ is related (Choy and Baxter 1987) to the three-spin correlation $\langle abc \rangle$ involving the corner spins from the elementary plaquette. The relationship is

$$\langle \sigma \rangle = M_\sigma = \left(\frac{\tanh 4K_1}{2} + \tanh 2K_1 \right) \langle a \rangle + \left(\frac{\tanh 4K_1}{2} - \tanh 2K_1 \right) \langle abc \rangle \tag{16}$$

where

$$\langle abc \rangle = \frac{F(x, y)}{G(x, y)} \langle a \rangle \tag{17}$$

and

$$F(x, y) = (x^4+1)^2(x^2-1)^2 + 2x^2y^2(x^4+1)(x^2-1)^2 + 2(x^2+1)^2(x^4+1)^2$$
$$- 2x^2y^2(x^2+1)^4 - 2\Delta(x, y)(x^2+1)^2(x^4+1) \tag{18}$$

$$G(x, y) = [(x^4+1)^2 - 4x^4y^2](x^2-1)^2 \tag{19}$$

$$\Delta(x, y) = [(1-x^4)^2 + 4x^4(1-y^2)^2]^{1/2}. \tag{20}$$

The evaluation of the remaining three-spin correlation entering (15) is straightforward. Using identity (12) and the symmetry of the correlation functions

$$\langle abc \rangle = \langle bcd \rangle = \langle cda \rangle = \langle dac \rangle \tag{21}$$

we find

$$\langle ab\sigma \rangle = \langle ab \tanh[K_1(a+b+c+d)] \rangle = \frac{\tanh 4K_1}{2}(\langle a \rangle + \langle abc \rangle). \tag{22}$$

Finally, the explicit expression for the mean spontaneous magnetization of the threefold coordinated sites on the pentagonal lattice is

$$\langle s_1 \rangle = \left[\frac{(A+C)\tanh 4K_1}{2} + A\tanh 2K_1 + 2B \right.$$
$$\left. + \left(\frac{(A+C)\tanh 4K_1}{2} - A\tanh 2K_1 \right) \frac{F(x, y)}{G(x, y)} \right] M_0. \tag{23}$$

Instead of a graphic presentation for the magnetization $M_0(T)$ and $M_s(T)$ we provide tables 1 and 2 with values of the magnetization for several reduced temperatures.

The differences between the values of the magnetization between sites with threefold and fourfold coordination are very small to be perceptible on a graph. As expected, the sites with lower coordination have weaker magnetization.

In conclusion, we have demonstrated that the Ising model on a pentagonal lattice with nearest-neighbour interactions characterized by two parameters is equivalent to the Ising model on the Union-Jack lattice. Exact expressions were obtained which permit calculation of the critical temperature and the magnetization of both types of lattice sites. The expressions correctly reproduce the known results in two limiting cases: $J' \to \infty$ when the pentagonal lattice is reduced to the square lattice, and $J' \to 0$ when it is reduced to a decorated square lattice. As expected the model falls in the same universality class with the nearest-neighbour Ising model on the square lattice. One may expect that the more general case of a pentagonal lattice with greater number of nearest-neighbour interaction parameters will also be exactly solvable because the necessary ingredients in the form of exact results for the anisotropic Union-Jack lattice are available in the literature (Baxter and Choy 1989, Lin and Wu 1989). Another possibility is to have as an elementary plaquette a square with a different motif. For example, if both internal spins in all plaquettes of the considered pentagonal lattice are substituted with triangles of spins connected between themselves at one identical site and additionally connected with their remaining sites with a pair of neighbouring corners of the underlying square lattice, we obtain a bowtie motif. Ordering the squares in the same manner as we did in figure 1, we have a lattice with triangles and septagons. Applying the star–triangle and dedecoration transformations, the Ising model on such a lattice can be exactly mapped onto the pentagonal lattice considered in this paper.

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